

D-brane in R-R Field Background

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Abstract

The purpose of this paper is to understand the low energy effective theory of a Dp -brane in the background of a large constant R-R $(p-1)$ -form field. We start with the M5-brane theory in large C-field background. The C -field background defines a 3-dimensional volume form on an M5-brane, and it is known that the low energy M5-brane theory can be described as a Nambu-Poisson gauge theory with the volume-preserving diffeomorphism symmetry (VPD). Via a double dimensional reduction we obtain a D4-brane in R-R 3-form field background. This theory has both the usual $U(1)$ gauge symmetry and the new symmetry of VPD. We find that the gauge potential for VPD is electric-magnetic dual to the $U(1)$ gauge potential, sharing the same physical degrees of freedom. The result can be generalized to Dp -branes.

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1 Introduction and Motivation

Low energy effective descriptions of D-branes and M-branes have played a crucial role in our understanding of string theory. They allow us to study a wide variety of subjects from AdS/CFT duality to brane world models. The two basic descriptions of D-branes are the Dirac-Born-Infeld (DBI) theory [1] and the super Yang-Mills (YM) theory [2]. Later it was realized that D-branes in large NS-NS B -field background should be described by gauge theories on noncommutative space [3–5]. The description of M5-branes was a challenging problem because of the self duality condition on gauge fields [6–8]. A covariant DBI-like action for a single M5-brane was first given in [7]. More recently, as an analogue of noncommutative D-branes in B -field background, the low energy effective theory for a single M5-brane in large C -field background was also found [9, 10]. The latter was actually derived from the Bagger-Lambert-Gustavsson model [11, 12] for multiple M2-branes. The understanding of branes often also help us understanding other branes.

The purpose of this paper is to construct new models for D-branes in R-R field backgrounds, to complete our understanding of D-branes in background fields. From the viewpoint of DBI theory or the YM theory, one can describe an R-R field background $A^{(p+1-2n)}$ simply by adding this term

$$\int A^{(p+1-2n)} F^n \tag{1}$$

to the D-brane action, where F is the $U(1)$ field strength. Why do we need to look for other descriptions? The answer is similar to why we need noncommutative gauge theories for D-branes in B -field background. The effect of a B -field can be incorporated into a D-brane action by simply replacing the field strength F by $B + F$. However, in the Seiberg-Witten limit [5], the noncommutative gauge theory is a better approximation than the result of replacing F by $B + F$ in the YM theory. Roughly speaking, when the B -field is large enough, higher derivative terms that would normally be ignored in a low energy effective theory can no longer be ignored if it is multiplied by a certain power of B . We would like to understand analogous effects of R-R fields on D-branes.

This problem has been studied in various aspects via different approaches. In [13], it was shown how R-R background potential influences D-brane dynamics in a way consistent with S-duality, so that Moyal deformation can be induced by R-R potential as well as the NS-NS B -field background. In [14], it was shown that the anti-commutation relation of fermionic fields can be modified by a graviphoton background, and the result was generalized to generic R-R backgrounds in [15]. In this paper we take yet another approach and find that a generalized Nambu-Poisson structure is induced by the R-R background,

characterizing a new gauge symmetry – the volume-preserving diffeomorphism – on the D-brane.

The noncommutativity on a D-brane due to a B -field background can roughly be understood as the effect due to the coupling of the B -field to an open string ending on the D-brane. Similarly, an R-R $(k + 1)$ -form gauge potential couples to a Dk -brane ending on a Dp -brane, and interaction of excitations on a Dp -brane mediated by a Dk -brane would be influenced by the R-R field background. Sufficiently strong R-R backgrounds can thus turn on new interactions usually ignored in a low energy effective theory.

Instead of computing directly the dynamics of D-branes ending on D-branes, our strategy is to fully utilize string dualities. The starting point is the above-mentioned new M5-brane theory [9, 10] in a large C -field background. The C -field background defines a 3-dimensional volume form, and the M5-brane theory is a gauge theory of diffeomorphisms preserving this volume form. We will refer to this theory as the Nambu-Poisson (NP) M5-brane theory for the gauge algebra is defined through the Nambu-Poisson bracket [16]. Various calculations [17] suggest that Nambu-Poisson bracket appears in a C -field background for open membranes in the same fashion that Moyal bracket appears in a B -field background for open strings.

An M5-brane is related to a D4-brane through double dimensional reduction (DDR), which means the simultaneous compactification of a direction in the target space and a direction on an M5-brane. The C -field in M theory leads to either a 3-form R-R gauge potential and/or a 2-form NS-NS B -field in the type IIA theory after compactification, depending on the direction of the compactified circle. In [10], the compactified circle is chosen such that the C -field background reduces to a B -field background, and the NP M5-brane theory reduces to the Poisson limit of the noncommutative gauge theory of a D4-brane.¹ In particular, the Nambu-Poisson bracket in the NP M5-brane theory reduces to the Poisson bracket. This can be viewed as an evidence for the validity of the NP M5-brane theory. Another evidence was obtained by examining the self dual string solutions corresponding to an M2-brane ending on an M5-brane [18]. A short review of the NP M5-brane was given in [19].

In this article we will carry out DDR in another direction so that the C -field background is reinterpreted as a constant R-R 3-form gauge potential. We will use the same symbol C to refer to both the M theory 3-form gauge potential and the 3-form R-R potential in type IIA string theory. The first goal of this paper is to understand the

¹ The Poisson limit of a noncommutative algebra refers to the approximation of the Moyal product by the leading order correction to the commutative product. In this limit the commutator of Moyal product reduces to the Poisson bracket.

D4-brane theory in a constant C -field background. It is expected that the geometry of this theory is equipped with a 3-bracket structure [20, 21].

The DDR of the NP M5-brane to D4-brane is highly nontrivial. The gauge symmetry of an NP M5-brane is the volume-preserving diffeomorphisms (VPD). Since the C -field background is parallel to the D4-brane, it is natural to expect that the D4-brane inherits the VPD symmetry. However, the massless spectrum of a D4-brane (a $U(1)$ gauge potential A , 5 scalars ϕ and their fermionic superpartners) does not include a 2-form gauge potential for the VPD gauge symmetry. We will show in the following that, interestingly, the 2-form gauge potential is dual to the 1-form $U(1)$ potential A , sharing the same physical degrees of freedom. While the VPD algebra is non-Abelian, the mathematical description of this duality is not straightforward at all.

The electric-magnetic duality between $U(1)$ gauge theory and VPD can be understood physically as follows. The endpoint of a fundamental string is an electric charge on a D4-brane, and the massless fluctuation of an open string in the longitudinal directions of the D4-brane constitutes the $U(1)$ gauge potential. From the M5-brane's viewpoint, the VPD gauge potential comes from the massless excitations of M2-branes ending on an M5-brane. It is a 2-form potential because the boundary of an M2-brane is a string. Therefore, from the D4-brane's viewpoint, the VPD gauge potential is associated with the boundary of a D2-brane, which is interpreted as the magnetic charge on the D4-brane, and so we expect the electric-magnetic duality between the $U(1)$ symmetry and VPD.

The interpretation above can be easily generalized to other Dp -branes. The endpoint of a fundamental string is an electric charge on the Dp -brane. The magnetic charge is then the boundary of a $D(p-2)$ -brane ending on the Dp -brane. Massless fluctuations of the $D(p-2)$ -brane in the longitudinal directions give rise to the $(p-2)$ -form potential for the VPD of the $(p-1)$ dimensional volume defined by an R-R $(p-1)$ -form background. The corresponding $(p-1)$ -form field strength is then dual to the 2-form field strength of the $U(1)$ symmetry. The second goal of this paper is to construct gauge theories describing a single Dp -brane in constant R-R $(p-1)$ -form field background. The results give us hints about Dp -brane theories in more general backgrounds of R-R fields. We leave this topic for future study.

The plan of this paper is as follows. We give a brief review of the NP M5-brane theory in Sec. 2. In Sec. 3 we derive the D4-brane action in large C -field background from the NP M5-brane action via double dimensional reduction, and in Sec. 4 we define 2-form field strengths F such that they are not only invariant under the $U(1)$ gauge transformations but also covariant under the VPD. We study the 0-th order and 1st order terms of the D4-brane action in the perturbative expansion in Sec. 5 to show how

the new action differs from Maxwell's action. For simplicity, matter fields are ignored in the calculation except in Sec. 6. We generalize the gauge field theory to multiple Dp-branes for generic p in Sec. 7. Finally we conclude in Sec. 8.

2 Review of Nambu-Poisson M5-brane Theory

The worldvolume theory of the M5-brane has the field content of a self-dual 2-form gauge potential $(b_{\dot{\mu}\dot{\nu}}, b_{\dot{\mu}\nu})$, 5 scalars (X^i) and the dimensional reduction of an 11 dimensional Majorana fermion (Ψ) .² The world volume coordinates will be denoted as $\{x^\mu, y^{\dot{\mu}}\} = \{x^0, x^1, x^2, y^{\dot{1}}, y^{\dot{2}}, y^{\dot{3}}\}$. In a C -field background the M5-brane action should respect the worldvolume translational symmetry, the global $SO(2, 1) \times SO(3)$ rotation symmetry, the gauge symmetry of volume-preserving diffeomorphisms and the 6D $\mathcal{N} = (2, 0)$ supersymmetry. In the limit $\epsilon \rightarrow 0$ [24] with

$$\ell_P \sim \epsilon^{1/3}, \quad g_{\mu\nu} \sim 1, \quad g_{\dot{\mu}\dot{\nu}} \sim \epsilon, \quad C_{\dot{\mu}\dot{\nu}\dot{\lambda}} \sim 1, \quad (\mu, \nu = 0, 1, 2 \text{ and } \dot{\mu}, \dot{\nu}, \dot{\lambda} = \dot{1}, \dot{2}, \dot{3}) \quad (2)$$

a good approximation of the M5-brane in C -field background is given by the action [10]³

$$S = S_X + S_\Psi + S_{gauge}, \quad S_{gauge} = S_{\mathcal{H}^2} + S_{CS}, \quad (3)$$

where⁴

$$S_X = \int d^3x d^3y \left[-\frac{1}{2}(\mathcal{D}_\mu X^i)^2 - \frac{1}{2}(\mathcal{D}_{\dot{\lambda}} X^i)^2 - \frac{1}{2g^2} - \frac{g^4}{4}\{X^{\dot{\mu}}, X^i, X^j\}^2 - \frac{g^4}{12}\{X^i, X^j, X^k\}^2 \right], \quad (4)$$

$$S_\Psi = \int d^3x d^3y \left[\frac{i}{2}\bar{\Psi}\Gamma^\mu \mathcal{D}_\mu \Psi + \frac{i}{2}\bar{\Psi}\Gamma^{\dot{\rho}} \mathcal{D}_{\dot{\rho}} \Psi + \frac{ig^2}{2}\bar{\Psi}\Gamma_{\dot{\mu}\dot{i}}\{X^{\dot{\mu}}, X^i, \Psi\} - \frac{ig^2}{4}\bar{\Psi}\Gamma_{ij}\Gamma_{\dot{1}\dot{2}\dot{3}}\{X^i, X^j, \Psi\} \right], \quad (5)$$

$$S_{\mathcal{H}^2} = \int d^3x d^3y \left[-\frac{1}{12}\mathcal{H}_{\dot{\mu}\dot{\nu}\dot{\rho}}^2 - \frac{1}{4}\mathcal{H}_{\lambda\dot{\mu}\dot{\nu}}^2 \right], \quad (6)$$

$$S_{CS} = \int d^3x d^3y \epsilon^{\mu\nu\lambda} \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} \left[-\frac{1}{2}\partial_{\dot{\mu}} b_{\mu\dot{\nu}} \partial_{\nu} b_{\lambda\dot{\lambda}} + \frac{g}{6}\partial_{\dot{\mu}} b_{\nu\dot{\nu}} \epsilon^{\dot{\rho}\dot{\sigma}\dot{\tau}} \partial_{\dot{\sigma}} b_{\lambda\dot{\rho}} (\partial_{\dot{\lambda}} b_{\mu\dot{\tau}} - \partial_{\dot{\tau}} b_{\mu\dot{\lambda}}) \right]. \quad (7)$$

² The fermion Ψ is chiral, i.e., $\Gamma^7 \Psi = -\Psi$, where Γ^7 is defined by $\Gamma^7 \equiv \Gamma^{012} \Gamma^{\dot{1}\dot{2}\dot{3}}$.

³ This action was first derived from the Bagger-Lambert action [11] with the Lie 3-algebra chosen to be the Nambu-Poisson algebra [9, 10].

⁴ Ψ here was denoted by Ψ' in [10].

In the above we use the notation

$$X^{\dot{\mu}}(y) \equiv \frac{y^{\dot{\mu}}}{g} + \frac{1}{2}\epsilon^{\dot{\mu}\kappa\dot{\lambda}}b_{\kappa\dot{\lambda}}(y), \quad (8)$$

$$\{A, B, C\} \equiv \epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}\partial_{\dot{\mu}}A\partial_{\dot{\nu}}B\partial_{\dot{\rho}}C. \quad (9)$$

The overall coefficient of the action S has been scaled to 1 by rescaling the fields and worldvolume coordinates.

For the matter fields X^i, Ψ , the covariant derivatives are defined by ⁵

$$\mathcal{D}_{\mu}\Phi \equiv \partial_{\mu}\Phi - g\{b_{\mu\nu}, y^{\nu}, \Phi\}, \quad (10)$$

$$\mathcal{D}_{\dot{\mu}}\Phi \equiv \frac{g^2}{2}\epsilon_{\dot{\mu}\dot{\nu}\dot{\rho}}\{X^{\dot{\nu}}, X^{\dot{\rho}}, \Phi\}. \quad (11)$$

The definition of the 3-form field strength as

$$H_{\lambda\dot{\mu}\dot{\nu}} = \partial_{\lambda}b_{\dot{\mu}\dot{\nu}} - \partial_{\dot{\mu}}b_{\lambda\dot{\nu}} + \partial_{\dot{\nu}}b_{\lambda\dot{\mu}}, \quad (12)$$

$$H_{\dot{\lambda}\dot{\mu}\dot{\nu}} = \partial_{\dot{\lambda}}b_{\dot{\mu}\dot{\nu}} + \partial_{\dot{\mu}}b_{\dot{\nu}\dot{\lambda}} + \partial_{\dot{\nu}}b_{\dot{\lambda}\dot{\mu}} \quad (13)$$

is no longer covariant under the non-Abelian gauge transformations. The covariant 3-form field strengths \mathcal{H} should be defined as

$$\begin{aligned} \mathcal{H}_{\lambda\dot{\mu}\dot{\nu}} &= \epsilon_{\dot{\mu}\dot{\nu}\dot{\lambda}}\mathcal{D}_{\lambda}X^{\dot{\lambda}} \\ &= H_{\lambda\dot{\mu}\dot{\nu}} - g\epsilon^{\dot{\sigma}\dot{\tau}\dot{\rho}}(\partial_{\dot{\sigma}}b_{\lambda\dot{\tau}})\partial_{\dot{\rho}}b_{\dot{\mu}\dot{\nu}}, \end{aligned} \quad (14)$$

$$\begin{aligned} \mathcal{H}_{\dot{1}\dot{2}\dot{3}} &= g^2\{X^{\dot{1}}, X^{\dot{2}}, X^{\dot{3}}\} - \frac{1}{g} \\ &= H_{\dot{1}\dot{2}\dot{3}} + \frac{g}{2}(\partial_{\dot{\mu}}b^{\dot{\mu}}\partial_{\dot{\nu}}b^{\dot{\nu}} - \partial_{\dot{\mu}}b^{\dot{\nu}}\partial_{\dot{\nu}}b^{\dot{\mu}}) + g^2\{b^{\dot{1}}, b^{\dot{2}}, b^{\dot{3}}\}. \end{aligned} \quad (15)$$

The fundamental fields transform under the gauge transformation as

$$\delta_{\Lambda}\Phi = g\kappa^{\dot{\rho}}\partial_{\dot{\rho}}\Phi \quad (\Phi = X^i, \Psi), \quad (16)$$

$$\delta_{\Lambda}b_{\kappa\dot{\lambda}} = \partial_{\kappa}\Lambda_{\dot{\lambda}} - \partial_{\dot{\lambda}}\Lambda_{\kappa} + g\kappa^{\dot{\rho}}\partial_{\dot{\rho}}b_{\kappa\dot{\lambda}}, \quad (17)$$

$$\delta_{\Lambda}b_{\lambda\dot{\sigma}} = \partial_{\lambda}\Lambda_{\dot{\sigma}} - \partial_{\dot{\sigma}}\Lambda_{\lambda} + g\kappa^{\dot{\tau}}\partial_{\dot{\tau}}b_{\lambda\dot{\sigma}} + g(\partial_{\dot{\sigma}}\kappa^{\dot{\tau}})b_{\lambda\dot{\tau}}, \quad (18)$$

where

$$\kappa^{\dot{\lambda}} \equiv \epsilon^{\dot{\lambda}\dot{\mu}\dot{\nu}}\partial_{\dot{\mu}}\Lambda_{\dot{\nu}}(x, y). \quad (19)$$

The field strengths \mathcal{H} transform like Φ .

⁵ Here and below we use Φ to represent both matter fields X^i, Ψ .

The gauge transformations can be more concisely expressed in terms of the new variables $b^{\dot{\mu}}, B_{\mu}^{\dot{\mu}}$

$$b^{\dot{\mu}} \equiv \frac{1}{2} \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} b_{\dot{\nu}\dot{\lambda}}, \quad (20)$$

$$B_{\mu}^{\dot{\mu}} \equiv \epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} \partial_{\dot{\nu}} b_{\mu\dot{\lambda}} \quad (21)$$

for the gauge fields as

$$\delta_{\Lambda} b^{\dot{\mu}} = \kappa^{\dot{\mu}} + g \kappa^{\dot{\nu}} \partial_{\dot{\nu}} b^{\dot{\mu}}, \quad (22)$$

$$\delta_{\Lambda} B_{\mu}^{\dot{\mu}} = \partial_{\mu} \kappa^{\dot{\mu}} + g \kappa^{\dot{\nu}} \partial_{\dot{\nu}} B_{\mu}^{\dot{\mu}} - g (\partial_{\dot{\nu}} \kappa^{\dot{\mu}}) B_{\mu}^{\dot{\nu}}. \quad (23)$$

In terms of $B_{\mu}^{\dot{\nu}}$, the covariant derivative \mathcal{D}_{μ} acts as

$$\mathcal{D}_{\mu} \Phi = \partial_{\mu} \Phi - g B_{\mu}^{\dot{\mu}} \partial_{\dot{\mu}} \Phi. \quad (24)$$

Remarkably, the field $b_{\mu\dot{\mu}}$ appears in the action only through the variable $B_{\mu}^{\dot{\mu}}$.

Another feature of the gauge transformations is that, in terms of $X^i, \Psi, b^{\dot{\mu}}$ and $B_{\mu}^{\dot{\mu}}$, all gauge transformations can be expressed solely in terms of $\kappa^{\dot{\mu}}$, without referring to $\Lambda_{\dot{\mu}}$, as long as one keeps in mind the constraint

$$\partial_{\dot{\mu}} \kappa^{\dot{\mu}} = 0. \quad (25)$$

This gauge transformation can be naturally interpreted as volume-preserving diffeomorphism (VPD)

$$\delta y^{\dot{\mu}} = g \kappa^{\dot{\mu}}, \quad \text{with} \quad \partial_{\dot{\mu}} \kappa^{\dot{\mu}} = 0. \quad (26)$$

The field $b^{\dot{\mu}}$ is then interpreted as the gauge potential for the VPD in the 3 dimensional space picked by the C -field background.

The M5-brane theory is also invariant under the supersymmetry transformations $\delta_{\chi}^{(1)}, \delta_{\epsilon}^{(2)}$. We have

$$\delta_{\chi}^{(1)} \Psi = \chi, \quad \delta_{\chi}^{(1)} X^i = \delta_{\chi}^{(1)} b_{\dot{\mu}\dot{\nu}} = \delta_{\chi}^{(1)} b_{\mu\dot{\nu}} = 0, \quad (27)$$

and ⁶

$$\delta_{\epsilon}^{(2)} X^i = i \bar{\epsilon} \Gamma^i \Psi, \quad (28)$$

$$\begin{aligned} \delta_{\epsilon}^{(2)} \Psi &= \mathcal{D}_{\mu} X^i \Gamma^{\mu} \Gamma^i \epsilon + \mathcal{D}_{\dot{\mu}} X^i \Gamma^{\dot{\mu}} \Gamma^i \epsilon \\ &\quad - \frac{1}{2} \mathcal{H}_{\mu\dot{\nu}\dot{\rho}} \Gamma^{\mu} \Gamma^{\dot{\nu}\dot{\rho}} \epsilon - \frac{1}{g} (1 + g \mathcal{H}_{\dot{1}\dot{2}\dot{3}}) \Gamma_{\dot{1}\dot{2}\dot{3}} \epsilon \\ &\quad - \frac{g^2}{2} \{X^{\dot{\mu}}, X^i, X^j\} \Gamma^{\dot{\mu}} \Gamma^{ij} \epsilon + \frac{g^2}{6} \{X^i, X^j, X^k\} \Gamma^{ijk} \Gamma^{\dot{1}\dot{2}\dot{3}} \epsilon, \end{aligned} \quad (29)$$

$$\delta_{\epsilon}^{(2)} b_{\dot{\mu}\dot{\nu}} = -i (\bar{\epsilon} \Gamma_{\dot{\mu}\dot{\nu}} \Psi), \quad (30)$$

$$\delta_{\epsilon}^{(2)} b_{\mu\dot{\nu}} = -i (1 + g \mathcal{H}_{\dot{1}\dot{2}\dot{3}}) \bar{\epsilon} \Gamma_{\mu} \Gamma_{\dot{\nu}} \Psi + i g (\bar{\epsilon} \Gamma_{\mu} \Gamma^i \Gamma_{\dot{1}\dot{2}\dot{3}} \Psi) \partial_{\dot{\nu}} X^i. \quad (31)$$

⁶ ϵ here was denoted by ϵ' in [10].

The SUSY transformation parameters χ, ϵ can be conveniently denoted as an 11D Majorana spinor satisfying the 6D chirality condition

$$\Gamma^7 \chi = \chi, \quad \Gamma^7 \epsilon = \epsilon. \quad (32)$$

They are both nonlinear SUSY transformations, but a superposition of the two,

$$\delta_\chi^{(1)} + g\delta_\epsilon^{(2)} \quad \text{with} \quad \chi = \Gamma^{i23}\epsilon, \quad (33)$$

defines a linear SUSY transformation.

3 D4-Brane via Double Dimensional Reduction

To carry out the double dimensional reduction (DDR) for the M5-brane along the x^2 -direction, we set

$$x^2 \sim x^2 + 2\pi R, \quad (34)$$

and let all other fields to be independent of x^2 . As a result we can set ∂_2 to zero when it acts on any field. Here R is the radius of the circle of compactification and we should take $R \ll 1$ such that the 6 dimensional field theory on M5 reduces to a 5 dimensional field theory for D4. Since the NP M5-brane action (3) is a good low energy effective theory in the limit (2), the 5 dimensional field theory is a good low energy effective description of a D4-brane in the limit $\epsilon \rightarrow 0$ for

$$\ell_s \sim \epsilon^{1/2}, \quad g_s \sim \epsilon^{-1/2}, \quad g_{\alpha\beta} \sim 1, \quad g_{\mu\nu} \sim \epsilon, \quad C_{\mu\nu\lambda} \sim 1, \quad (35)$$

with

$$g_s \ell_s \ll 1, \quad (36)$$

from the perspective of the type IIA theory. The indices $\alpha, \beta = 0, 1$ are used to distinguish from the M5-brane indices $\mu, \nu = 0, 1, 2$.

Note that in the limit (2) the C -field component $C_{012} \sim \epsilon^{-1}$. As a result the B -field component $B_{01} \sim \epsilon^{-1}$ and the noncommutative parameter $\theta^{01} \sim B^{-1} \sim \epsilon$ vanishes in the limit $\epsilon \rightarrow 0$. However, the combination $2\pi\alpha'B$ is finite in the limit, and thus the D4-brane is not only in a C -field background but also in the B -field background. Using the nonlinear self-dual relation derived in [6], we can express C_{012} in terms of C_{i23} , and then the B -field background is given by

$$2\pi\alpha'B_{01} = \frac{C_{i23}}{2\pi}. \quad (37)$$

In the convention (normalization of the worldvolume coordinates) of [10], we have

$$C_{i23} = \frac{1}{g^2} \quad \Rightarrow \quad 2\pi\alpha'B_{01} = \frac{1}{2\pi g^2}. \quad (38)$$

3.1 D4-brane Action for the Gauge Fields

For simplicity let us ignore the matter fields for the time being, and focus on the gauge field part of the action S_{gauge} . We will give the full action including matter fields later in Sec. 6. The result of DDR on S_{gauge} is

$$S_{gauge}^{(1)}[b^{\dot{\mu}}, a_{\dot{\mu}}, b_{\alpha\dot{\mu}}] = \int d^2x d^3y \left\{ -\frac{1}{2}\mathcal{H}_{\dot{1}\dot{2}\dot{3}}^2 - \frac{1}{4}\mathcal{H}_{2\dot{\mu}\dot{\nu}}^2 - \frac{1}{4}\mathcal{H}_{\alpha\dot{\mu}\dot{\nu}}^2 \right. \\ \left. + \epsilon^{\alpha\beta}\epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}\partial_{\beta}a_{\dot{\rho}}\partial_{\dot{\mu}}b_{\alpha\dot{\nu}} + \frac{g}{2}\epsilon^{\alpha\beta}\epsilon_{\dot{\mu}\dot{\nu}\dot{\rho}}\epsilon^{\dot{\mu}\dot{\delta}\dot{\tau}}\epsilon^{\dot{\nu}\dot{\sigma}\dot{\lambda}}\epsilon^{\dot{\rho}\dot{\eta}\dot{\xi}}\partial_{\dot{\delta}}b_{\alpha\dot{\tau}}\partial_{\dot{\sigma}}b_{\beta\dot{\lambda}}\partial_{\dot{\eta}}a_{\dot{\xi}} \right\}, \quad (39)$$

where

$$a_{\dot{\mu}} \equiv b_{\dot{\mu}2}. \quad (40)$$

Apparently we should identify $a_{\dot{\mu}}$ as components of the one-form potential on the D4-brane. In terms of the field strength

$$F_{\dot{\mu}\dot{\nu}} \equiv \partial_{\dot{\mu}}a_{\dot{\nu}} - \partial_{\dot{\nu}}a_{\dot{\mu}}, \quad (41)$$

we can rewrite $\mathcal{H}_{2\dot{\mu}\dot{\nu}}$ as

$$\mathcal{H}_{2\dot{\mu}\dot{\nu}} = F_{\dot{\mu}\dot{\nu}} + \frac{g}{2}\epsilon_{\dot{\mu}\dot{\nu}\dot{\lambda}}\epsilon^{\dot{\sigma}\dot{\rho}\dot{\tau}}\partial_{\dot{\sigma}}b^{\dot{\lambda}}F_{\dot{\rho}\dot{\tau}}. \quad (42)$$

In the above we see that part of the two-form potential b on the M5-brane transforms into part of the one-form potential a on D4. However, in order to interpret this action as a D4-brane action, we still need to identify the rest of the components a_{α} of the one-form gauge potential, and to re-interpret $b_{\alpha\dot{\mu}}$ and $b_{\dot{\mu}\dot{\nu}}$ from the D4-brane viewpoint. We expect that the $U(1)$ gauge symmetry on the D4-brane has its origin in the gauge transformations (17), (18) on the M5-brane. The gauge transformation parameter Λ_2 shall be identified with the $U(1)$ gauge transformation parameter. This is consistent with the identification of $a_{\dot{\mu}}$ with $b_{\dot{\mu}2}$. The gauge symmetry parametrized by $\Lambda_{\dot{\mu}}$, i.e., the VPD, is also still present on the D4-brane.

3.2 Duality Transformation

In order to understand the physical meaning of the action (39), we try to simplify the action by integrating out the remaining components of the 2-form gauge field b as much as possible, since there is no 2-form gauge potential in the usual description of a D4-brane.

First we note that the action (39) depends on $b_{\alpha\dot{\mu}}$ only through the variable $B_{\alpha}^{\dot{\mu}}$ (21). In terms of $B_{\alpha}^{\dot{\mu}}$, we have

$$\mathcal{H}_{\alpha\dot{\mu}\dot{\nu}} = \epsilon_{\dot{\mu}\dot{\nu}\dot{\lambda}}(\partial_{\alpha}b^{\dot{\lambda}} - V_{\sigma}^{\dot{\lambda}}B_{\alpha}^{\dot{\sigma}}), \quad (43)$$

where

$$V_{\dot{\nu}}^{\dot{\mu}} \equiv \delta_{\dot{\nu}}^{\dot{\mu}} + g \partial_{\dot{\nu}} b^{\dot{\mu}}. \quad (44)$$

Hence we can rewrite the action (39) as

$$\begin{aligned} S^{(2)}[b^{\dot{\mu}}, a_{\dot{\mu}}, B_{\alpha}^{\dot{\mu}}] &= \int d^2 x d^3 y \left\{ -\frac{1}{2} \mathcal{H}_{1\dot{2}\dot{3}}^2 - \frac{1}{4} \mathcal{H}_{2\dot{\mu}\dot{\nu}}^2 \right. \\ &\quad \left. - \frac{1}{2} (\partial_{\alpha} b^{\dot{\mu}} - V_{\dot{\sigma}}^{\dot{\mu}} B_{\alpha}^{\dot{\sigma}})^2 + \epsilon^{\alpha\beta} \partial_{\beta} a_{\dot{\mu}} B_{\alpha}^{\dot{\mu}} + \frac{g}{2} \epsilon^{\alpha\beta} F_{\dot{\mu}\dot{\nu}} B_{\alpha}^{\dot{\mu}} B_{\beta}^{\dot{\nu}} \right\}. \end{aligned} \quad (45)$$

It turns out that it is possible to extract the components a_{α} on the D4-brane by dualizing the field $B_{\alpha}^{\dot{\mu}}$. We can introduce the Lagrange multiplier $f_{\alpha\dot{\mu}}$ to rewrite the action (45) as

$$\begin{aligned} S^{(3)}[b^{\dot{\mu}}, a_{\dot{\mu}}, b_{\alpha\dot{\mu}}, \check{B}_{\alpha}^{\dot{\mu}}, f_{\beta\dot{\mu}}] &= \int d^2 x d^3 y \left\{ -\frac{1}{2} \mathcal{H}_{1\dot{2}\dot{3}}^2 - \frac{1}{4} \mathcal{H}_{2\dot{\mu}\dot{\nu}}^2 - \frac{1}{2} (\partial_{\alpha} b^{\dot{\mu}} - V_{\dot{\sigma}}^{\dot{\mu}} \check{B}_{\alpha}^{\dot{\sigma}})^2 \right. \\ &\quad \left. + \epsilon^{\alpha\beta} \partial_{\beta} a_{\dot{\mu}} \check{B}_{\alpha}^{\dot{\mu}} + \frac{g}{2} \epsilon^{\alpha\beta} F_{\dot{\mu}\dot{\nu}} \check{B}_{\alpha}^{\dot{\mu}} \check{B}_{\beta}^{\dot{\nu}} - \epsilon^{\alpha\beta} f_{\beta\dot{\mu}} [\check{B}_{\alpha}^{\dot{\mu}} - \epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}} \partial_{\dot{\nu}} b_{\alpha\dot{\rho}}] \right\}, \end{aligned} \quad (46)$$

where we used the notation \check{B} for a new variable independent of $b_{\alpha\dot{\mu}}$. If we integrate out the Lagrange multiplier $f_{\beta\dot{\mu}}$, we will get $\check{B}_{\alpha}^{\dot{\mu}} = B_{\alpha}^{\dot{\mu}}$, and the action above reduces back to (45).

Instead, we can integrate out $\check{B}_{\alpha}^{\dot{\mu}}$ and $b_{\alpha\dot{\mu}}$ to dualize the field $B_{\alpha}^{\dot{\mu}}$. First we integrate out $b_{\alpha\dot{\mu}}$, and find the constraint on $f_{\alpha\dot{\mu}}$

$$\epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} \partial_{\dot{\mu}} f_{\alpha\dot{\nu}} = 0. \quad (47)$$

It implies that, locally

$$f_{\alpha\dot{\mu}} = \partial_{\dot{\mu}} a_{\alpha} \quad (48)$$

for some potential a_{α} . Hence, after integrating out $b_{\alpha\dot{\mu}}$, we get

$$\begin{aligned} S^{(4)}[b^{\dot{\mu}}, a_{\dot{\mu}}, a_{\alpha}, \check{B}_{\alpha}^{\dot{\mu}}] &= \int d^2 x d^3 y \left\{ -\frac{1}{2} \mathcal{H}_{1\dot{2}\dot{3}}^2 - \frac{1}{4} \mathcal{H}_{2\dot{\mu}\dot{\nu}}^2 - \frac{1}{2} (\partial_{\alpha} b^{\dot{\mu}} - V_{\dot{\sigma}}^{\dot{\mu}} \check{B}_{\alpha}^{\dot{\sigma}})^2 \right. \\ &\quad \left. + \epsilon^{\alpha\beta} \partial_{\beta} a_{\dot{\mu}} \check{B}_{\alpha}^{\dot{\mu}} + \frac{g}{2} \epsilon^{\alpha\beta} F_{\dot{\mu}\dot{\nu}} \check{B}_{\alpha}^{\dot{\mu}} \check{B}_{\beta}^{\dot{\nu}} - \epsilon^{\alpha\beta} \partial_{\dot{\mu}} a_{\beta} \check{B}_{\alpha}^{\dot{\mu}} \right\}. \end{aligned} \quad (49)$$

The next step is to integrate out $\check{B}_{\alpha}^{\dot{\mu}}$ to get the dual action. Since the action is at most quadratic in $\check{B}_{\alpha}^{\dot{\mu}}$, the result of integrating out $\check{B}_{\alpha}^{\dot{\mu}}$ is the same as replacing $\check{B}_{\alpha}^{\dot{\mu}}$ by the solution to its equation of motion, which is a constraint

$$V_{\dot{\mu}}^{\dot{\nu}} (\partial^{\alpha} b_{\dot{\nu}} - V_{\dot{\nu}}^{\dot{\rho}} \check{B}_{\dot{\rho}}^{\alpha}) + \epsilon^{\alpha\beta} F_{\beta\dot{\mu}} + g \epsilon^{\alpha\beta} F_{\dot{\mu}\dot{\nu}} \check{B}_{\beta}^{\dot{\nu}} = 0. \quad (50)$$

The solution of $\check{B}_\alpha{}^\mu$, denoted as $\hat{B}_\alpha{}^\mu$, is given by

$$\hat{B}_\alpha{}^\mu \equiv (M^{-1})_{\alpha\beta}{}^{\mu\nu} (V_\nu{}^\sigma \partial^\beta b_\sigma + \epsilon^{\beta\gamma} F_{\gamma\nu}), \quad (51)$$

where

$$M_{\mu\nu}{}^{\alpha\beta} \equiv V_{\mu\rho} V_\nu{}^\rho \delta^{\alpha\beta} - g \epsilon^{\alpha\beta} F_{\mu\nu}, \quad (52)$$

and M^{-1} is defined by

$$(M^{-1})_{\gamma\alpha}{}^{\lambda\mu} M_{\mu\nu}{}^{\alpha\beta} = \delta^\lambda{}_\nu \delta_\gamma{}^\beta. \quad (53)$$

After integrating out $\check{B}_\alpha{}^\mu$, we get

$$\begin{aligned} S^{(5)}[b^\mu, a_\mu, a_\alpha] = & \int d^2x d^3y \left\{ -\frac{1}{2} \mathcal{H}_{123}^2 - \frac{1}{4} (F_{\nu\rho} + \frac{g}{2} \epsilon_{\mu\nu\rho} \epsilon^{\sigma\delta\tau} \partial_\sigma b^\mu F_{\delta\tau})^2 - \frac{1}{2} \partial_\alpha b^\mu \partial^\alpha b_\mu \right. \\ & \left. + \frac{1}{2} (\epsilon^{\alpha\gamma} F_{\gamma\mu} + V_\mu{}^\sigma \partial^\alpha b_\sigma) (M^{-1})_{\alpha\beta}{}^{\mu\nu} (\epsilon^{\beta\delta} F_{\delta\nu} + V_\nu{}^\lambda \partial^\beta b_\lambda) \right\}. \quad (54) \end{aligned}$$

At the quantum level, there is a one-loop contribution to the action when we integrate out $\check{B}_\alpha{}^\mu$. It is

$$\Delta S_{1-loop} = -\frac{\hbar}{2} \text{Tr}(\text{Log}(M_{\mu\nu}{}^{\alpha\beta})). \quad (55)$$

The action (54) is only remotely resembling the familiar Maxwell action for a $U(1)$ gauge theory we expect on the D4-brane. We can find terms resembling $F_{\mu\nu}^2$ and $F_{\alpha\beta}^2$, but the coefficients do not match. The term $F_{\alpha\beta}^2$ is missing. We still have the field b^μ which can not be easily integrated out because it has 2nd derivative terms in the action. It appears that we need to keep the field b^μ , which continues to play the role of the gauge potential for the gauge transformation parametrized by Λ_μ , but we need to identify its physical degrees of freedom in the D4-brane theory.

Having decided to keep the gauge transformations parametrized by Λ_μ as a new gauge symmetry in the D4-brane theory, we need to define covariant field strengths suitable for the gauge transformations.

4 Covariant Variables

4.1 Gauge Transformation

The gauge transformations of b^μ and $a_\mu = b_{\mu 2}$ are inherited from the NP M5-brane theory as

$$\delta_\Lambda b^\mu = \kappa^\mu + g \kappa^\nu \partial_\nu b^\mu, \quad (56)$$

$$\delta_\Lambda a_\mu = \partial_\mu \lambda + g(\kappa^\nu \partial_\nu a_\mu + a_\nu \partial_\mu \kappa^\nu), \quad (57)$$

where $\lambda \equiv \Lambda_2$.

The field a_α was introduced by hand and so its gauge transformation rule has to be solved from the requirement that the action $S^{(4)}$ (49) be invariant. For a quick derivation one needs to realize that the Chern-Simons term must be gauge invariant by itself. Plugging in the gauge transformation of $\check{B}_\alpha^{\dot{\mu}}$ ⁷ and b^μ , the gauge transformation of the CS term (after integration by part) is

$$\begin{aligned} & \delta_\Lambda(\epsilon^{\alpha\beta}\partial_\beta a_{\dot{\mu}}\check{B}_\alpha^{\dot{\mu}} + \frac{g}{2}\epsilon^{\alpha\beta}F_{\dot{\mu}\dot{\nu}}\check{B}_\alpha^{\dot{\mu}}\check{B}_\beta^{\dot{\nu}} - \epsilon^{\alpha\beta}\partial_{\dot{\mu}}a_\beta\check{B}_\alpha^{\dot{\mu}}) \\ &= \partial_{\dot{\mu}}\check{B}_\alpha^{\dot{\mu}}\epsilon^{\alpha\beta}[-\partial_\beta\lambda - g(\kappa^{\dot{\sigma}}\partial_{\dot{\sigma}}a_\beta + a_{\dot{\sigma}}\partial_\beta\kappa^{\dot{\sigma}}) + \delta a_\beta]. \end{aligned} \quad (58)$$

Hence we get

$$\delta_\Lambda a_\beta = \partial_\beta\lambda + g(\kappa^{\dot{\sigma}}\partial_{\dot{\sigma}}a_\beta + a_{\dot{\sigma}}\partial_\beta\kappa^{\dot{\sigma}}). \quad (59)$$

In our formulation of the self dual gauge field b , the components $b_{\mu\nu}$ do not explicitly show up in the action. Rather they appear when we solve the equations of motion for the rest of the components $b_{\dot{\mu}\dot{\nu}}$ and $b_{\mu\dot{\mu}}$. In [22, 23], the components $b_{\mu\nu}$ are used to explicitly exhibit the self duality of the gauge field, and their gauge transformation laws are given by

$$\delta_\Lambda b_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu + g[\kappa^{\dot{\rho}}(\partial_{\dot{\rho}}b_{\mu\nu}) + (\partial_\nu\kappa^{\dot{\rho}})b_{\mu\dot{\rho}} - (\partial_\mu\kappa^{\dot{\rho}})b_{\nu\dot{\rho}}]. \quad (60)$$

Identifying $b_{\beta 2}$ with a_β and setting $\partial_2 = 0$ for DDR, we get exactly the same gauge transformation rule as (59) with $\Lambda_2 = \lambda$.

We find that the gauge transformation of $a_{\dot{\mu}}$ (57) and that of a_α (59) are of the same form

$$\delta_\Lambda a_A = \partial_A\lambda + g(\kappa^{\dot{\nu}}\partial_{\dot{\nu}}a_A + a_{\dot{\nu}}\partial_A\kappa^{\dot{\nu}}). \quad (61)$$

For the convenience of the reader, let us also give here the gauge transformation of $V_{\dot{\nu}}^{\dot{\mu}}$, $M_{\dot{\mu}\dot{\nu}}^{\alpha\beta}$ and $\hat{B}_\alpha^{\dot{\mu}}$:

$$\delta_\Lambda V_{\dot{\nu}}^{\dot{\mu}} = g\kappa^{\dot{\lambda}}\partial_{\dot{\lambda}}V_{\dot{\nu}}^{\dot{\mu}} + g(\partial_{\dot{\nu}}\kappa^{\dot{\lambda}})V_{\dot{\lambda}}^{\dot{\mu}}, \quad (62)$$

$$\delta_\Lambda M_{\dot{\mu}\dot{\nu}}^{\alpha\beta} = g[\kappa^{\dot{\sigma}}\partial_{\dot{\sigma}}M_{\dot{\mu}\dot{\nu}}^{\alpha\beta} + (\partial_{\dot{\mu}}\kappa^{\dot{\sigma}})M_{\dot{\sigma}\dot{\nu}}^{\alpha\beta} + (\partial_{\dot{\nu}}\kappa^{\dot{\sigma}})M_{\dot{\mu}\dot{\sigma}}^{\alpha\beta}], \quad (63)$$

$$\delta_\Lambda \hat{B}_\alpha^{\dot{\mu}} = \partial_\alpha\kappa^{\dot{\mu}} + g(\kappa^{\dot{\nu}}\partial_{\dot{\nu}}\hat{B}_\alpha^{\dot{\mu}} - \hat{B}_\alpha^{\dot{\nu}}\partial_{\dot{\nu}}\kappa^{\dot{\mu}}). \quad (64)$$

⁷ The gauge transformation of $\check{B}_\alpha^{\dot{\mu}}$ should be the same as that of $B_\alpha^{\dot{\mu}}$.

4.2 Covariant Field Strengths

In the original NP M5-brane theory, we have the covariant field strengths ⁸

$$\mathcal{H}_{1\dot{2}\dot{3}} = \partial_{\dot{\mu}} b^{\dot{\mu}} + \frac{1}{2}g(\partial_{\dot{\nu}} b^{\dot{\nu}} \partial_{\dot{\rho}} b^{\dot{\rho}} - \partial_{\dot{\nu}} b^{\dot{\rho}} \partial_{\dot{\rho}} b^{\dot{\nu}}) + g^2\{b^{\dot{1}}, b^{\dot{2}}, b^{\dot{3}}\}, \quad (65)$$

$$\mathcal{F}_{\dot{\mu}\dot{\nu}} \equiv \mathcal{H}_{2\dot{\mu}\dot{\nu}} = F_{\dot{\mu}\dot{\nu}} + g[\partial_{\dot{\sigma}} b^{\dot{\sigma}} F_{\dot{\mu}\dot{\nu}} - \partial_{\dot{\mu}} b^{\dot{\sigma}} F_{\dot{\sigma}\dot{\nu}} - \partial_{\dot{\nu}} b^{\dot{\sigma}} F_{\dot{\mu}\dot{\sigma}}], \quad (66)$$

which survive the DDR. Here we have also rewritten $\mathcal{H}_{2\dot{\mu}\dot{\nu}}$, which was given above in (42), in a different but equivalent form.

The covariant version of $F_{\alpha\dot{\mu}}$ can be defined as

$$\mathcal{F}_{\alpha\dot{\mu}} \equiv \frac{1}{2}\epsilon_{\beta\alpha}\epsilon_{\dot{\mu}\dot{\nu}\dot{\lambda}}\mathcal{H}^{\beta\dot{\nu}\dot{\lambda}}. \quad (67)$$

This is motivated by the intuition that $\mathcal{F}_{\alpha\dot{\mu}}$ corresponds to $\mathcal{H}_{\alpha\dot{\mu}2}$ in the M5-brane theory, and we used the self duality condition of \mathcal{H} to write down the expression above. Replacing $B_{\alpha}^{\dot{\mu}}$ by the solution $\hat{B}_{\alpha}^{\dot{\mu}}$, we can rewrite $\mathcal{H}^{\beta\dot{\nu}\dot{\lambda}}$ (43) as a function of F_{AB} , $\partial_{\dot{\mu}} b^{\dot{\nu}}$ and $\hat{B}_{\alpha}^{\dot{\mu}}$. (That is, we avoided using $\partial_{\alpha} b^{\dot{\mu}}$ directly. The dependence on $\partial_{\alpha} b^{\dot{\mu}}$ only appears through $\hat{B}_{\alpha}^{\dot{\mu}}$.) As a result, we have

$$\mathcal{F}_{\alpha\dot{\mu}} = V^{-1}_{\dot{\mu}}{}^{\dot{\nu}}(F_{\alpha\dot{\nu}} + gF_{\dot{\nu}\dot{\sigma}}\hat{B}_{\alpha}^{\dot{\sigma}}). \quad (68)$$

This is also in agreement with the definition of $\mathcal{H}_{\mu\nu\dot{\mu}}$ defined in [22, 23].

By inspection, we can guess the covariant form of $F_{\alpha\beta}$. Together with the rest of the covariant field strengths of the $U(1)$ gauge field, we have

$$\begin{aligned} \mathcal{F}_{\dot{\mu}\dot{\nu}} &= F_{\dot{\mu}\dot{\nu}} + g[\partial_{\dot{\sigma}} b^{\dot{\sigma}} F_{\dot{\mu}\dot{\nu}} - \partial_{\dot{\mu}} b^{\dot{\sigma}} F_{\dot{\sigma}\dot{\nu}} - \partial_{\dot{\nu}} b^{\dot{\sigma}} F_{\dot{\mu}\dot{\sigma}}] \\ &= V_{\dot{\rho}}{}^{\dot{\rho}} F_{\dot{\mu}\dot{\nu}} + V_{\dot{\mu}}{}^{\dot{\rho}} F_{\dot{\nu}\dot{\rho}} + V_{\dot{\nu}}{}^{\dot{\rho}} F_{\dot{\rho}\dot{\mu}}, \end{aligned} \quad (69)$$

$$\mathcal{F}_{\alpha\dot{\mu}} = V^{-1}_{\dot{\mu}}{}^{\dot{\nu}}(F_{\alpha\dot{\nu}} + gF_{\dot{\nu}\dot{\sigma}}\hat{B}_{\alpha}^{\dot{\sigma}}), \quad (70)$$

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + g[-F_{\alpha\dot{\mu}}\hat{B}_{\beta}^{\dot{\mu}} - F_{\dot{\mu}\beta}\hat{B}_{\alpha}^{\dot{\mu}} + gF_{\dot{\mu}\dot{\nu}}\hat{B}_{\alpha}^{\dot{\mu}}\hat{B}_{\beta}^{\dot{\nu}}], \quad (71)$$

where

$$F_{AB} \equiv \partial_A a_B - \partial_B a_A. \quad (72)$$

Unlike $\mathcal{F}_{\dot{\mu}\dot{\nu}}$ and $\mathcal{F}_{\alpha\dot{\mu}}$, the components $\mathcal{F}_{\alpha\beta}$ can not be directly matched with the field $\mathcal{H}_{\alpha\beta 2}$ in the M5-brane theory, because the latter involves other fields that does not exist in the D4-brane theory.

⁸ A field Φ is covariant if its gauge transformation is $\delta_{\Lambda}\Phi = g\kappa^{\dot{\mu}}\partial_{\dot{\mu}}\Phi$.

4.3 D4-brane Action in Terms of Covariant Variables

Remarkably, in terms of the covariant field strengths, the action is simply

$$S'_{gauge}[b^\mu, a_A] = \int d^2x d^3y \left\{ -\frac{1}{2} \mathcal{H}_{1\dot{2}\dot{3}} \mathcal{H}^{1\dot{2}\dot{3}} - \frac{1}{4} \mathcal{F}_{\dot{\nu}\dot{\rho}} \mathcal{F}^{\dot{\nu}\dot{\rho}} + \frac{1}{2} \mathcal{F}_{\beta\dot{\mu}} \mathcal{F}^{\beta\dot{\mu}} + \frac{1}{2g} \epsilon^{\alpha\beta} \mathcal{F}_{\alpha\beta} \right\}. \quad (73)$$

The last term in the Lagrangian resembles the Wess-Zumino term for the C -field.

It appears that we are missing the kinetic term $\mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta}$ in the Lagrangian, and the coefficient of the term $\mathcal{F}_{\alpha\dot{\mu}} \mathcal{F}^{\alpha\dot{\mu}}$ is wrong. However, in the next section we will see that the missing kinetic term arises when we integrate out b^μ .

5 D4-brane Action Expanded

5.1 Zeroth Order

In this subsection we show that at the lowest order of g , the D4-brane action (73) agrees with the Maxwell action for a $U(1)$ gauge field in the ordinary D4-brane action. First we expand everything to the 1st order

$$\mathcal{H}_{1\dot{2}\dot{3}} = \partial_{\dot{\mu}} b^\mu + g \frac{1}{2} (\partial_{\dot{\nu}} b^\nu \partial_{\dot{\rho}} b^\rho - \partial_{\dot{\nu}} b^\rho \partial_{\dot{\rho}} b^\nu) + \mathcal{O}(g^2), \quad (74)$$

$$V^{-1}_{\dot{\mu}}{}^{\dot{\nu}} = \delta_{\dot{\mu}}{}^{\dot{\nu}} - g \partial_{\dot{\mu}} b^\nu + \mathcal{O}(g^2), \quad (75)$$

$$(M^{-1})^{\dot{\mu}\dot{\nu}}_{\alpha\beta} = \delta^{\dot{\mu}\dot{\nu}}_{\alpha\beta} - g [(\partial^{\dot{\mu}} b^\nu + \partial^{\dot{\nu}} b^\mu) \delta_{\alpha\beta} - \epsilon_{\alpha\beta} F^{\dot{\mu}\dot{\nu}}] + \mathcal{O}(g^2), \quad (76)$$

$$\begin{aligned} \hat{B}_\alpha{}^{\dot{\mu}} &= \partial_\alpha b^\mu + \epsilon_{\alpha\beta} F^{\beta\dot{\mu}} + g [-\partial^{\dot{\sigma}} b^\mu \partial_\alpha b_{\dot{\sigma}} - \partial^\mu b_{\dot{\sigma}} \epsilon_{\alpha\beta} F^{\beta\dot{\sigma}} - \partial_{\dot{\sigma}} b^\mu \epsilon_{\alpha\beta} F^{\beta\dot{\sigma}} \\ &\quad + \epsilon_{\alpha\beta} \partial^\beta b_{\dot{\sigma}} F^{\dot{\mu}\dot{\sigma}} + F_{\alpha\dot{\sigma}} F^{\dot{\mu}\dot{\sigma}}] + \mathcal{O}(g^2). \end{aligned} \quad (77)$$

$$\mathcal{F}_{\beta\dot{\mu}} = F_{\beta\dot{\mu}} + g (\partial_{\dot{\mu}} b^\sigma F_{\sigma\beta} + \partial_\beta b^\sigma F_{\dot{\mu}\sigma} + \epsilon_{\beta\gamma} F_{\dot{\mu}\sigma} F^{\gamma\sigma}) + \mathcal{O}(g^2) \quad (78)$$

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + g [-F_{\alpha\dot{\mu}} (\partial_\beta b^\mu + \epsilon_{\beta\gamma} F^{\gamma\dot{\mu}}) - F_{\dot{\mu}\beta} (\partial_\alpha b^\mu + \epsilon_{\alpha\gamma} F^{\gamma\dot{\mu}})] + \mathcal{O}(g^2). \quad (79)$$

To the lowest order of g , the last term in the Lagrangian (73) is

$$\begin{aligned} \frac{1}{2g} \epsilon^{\alpha\beta} \mathcal{F}_{\alpha\beta} &= \frac{1}{2g} \epsilon^{\alpha\beta} F_{\alpha\beta} + \frac{1}{2} \epsilon^{\alpha\beta} [-F_{\alpha\dot{\mu}} (\partial_\beta b^\mu + \epsilon_{\beta\gamma} F^{\gamma\dot{\mu}}) - F_{\dot{\mu}\beta} (\partial_\alpha b^\mu + \epsilon_{\alpha\gamma} F^{\gamma\dot{\mu}})] + \mathcal{O}(g) \\ &\simeq -\epsilon^{\alpha\beta} F_{\alpha\dot{\mu}} \partial_\beta b^\mu - F_{\alpha\dot{\mu}} F^{\alpha\dot{\mu}} + \mathcal{O}(g) \\ &\simeq \epsilon^{\alpha\beta} \partial_\beta a_\alpha \partial_{\dot{\mu}} b^\mu - F_{\alpha\dot{\mu}} F^{\alpha\dot{\mu}} + \mathcal{O}(g), \end{aligned} \quad (80)$$

up to total derivatives. To the 0-th order of g , the action (73) can now be expressed as

$$\begin{aligned} S'^{(0)}_{gauge}[b^\mu, a_A] &= \int d^2x d^3y \left\{ -\frac{1}{2} H_{1\dot{2}\dot{3}}^2 - \frac{1}{2} \epsilon^{\alpha\beta} F_{\alpha\beta} H_{1\dot{2}\dot{3}} - \frac{1}{4} F_{\dot{\mu}\dot{\nu}} F^{\dot{\mu}\dot{\nu}} - \frac{1}{2} F_{\alpha\dot{\mu}} F^{\alpha\dot{\mu}} \right\} \\ &= \int d^2x d^3y \left\{ -\frac{1}{2} (H_{1\dot{2}\dot{3}} + F_{01})^2 - \frac{1}{4} F_{AB} F^{AB} \right\}, \end{aligned} \quad (81)$$

where $H_{i\dot{2}\dot{3}} = \partial_{\dot{\mu}} b^{\dot{\mu}}$ and $A, B = (\dot{\mu}, \alpha)$. Note that $H_{i\dot{2}\dot{3}}$ is the only gauge invariant degree of freedom in the gauge potential $b^{\dot{\mu}}$ because there are two independent gauge transformation parameters.⁹ Furthermore there is no kinetic term for $b^{\dot{\mu}}$ and so we can integrate it out and then (81) becomes exactly the Maxwell action. Integrating out $b^{\dot{\mu}}$ is a duality transformation which imposes the identification

$$H_{i\dot{2}\dot{3}} = -F_{01}. \quad (82)$$

The physical degrees of freedom in $b^{\dot{\mu}}$ is transformed into that of a_{α} . Although $b^{\dot{\mu}}$ appears as new gauge potentials in the D4-brane theory, they share the same physical degrees of freedom with a_{α} .

5.2 First Order

The first order correction to the action (81) is

$$\begin{aligned} S'_{gauge}{}^{(1)}[b^{\dot{\mu}}, a_A] &= g \int d^2x d^3y \left\{ \left(-\frac{1}{2} H_{i\dot{2}\dot{3}}^2 + \frac{1}{2} \partial_{\dot{\mu}} b^{\dot{\nu}} \partial_{\dot{\nu}} b^{\dot{\mu}} \right) (H_{i\dot{2}\dot{3}} + F_{01}) \right. \\ &\quad + H_{i\dot{2}\dot{3}} \left(-\frac{1}{2} F_{\dot{\mu}\dot{\nu}} F^{\dot{\mu}\dot{\nu}} + \epsilon^{\alpha\beta} F_{\alpha\dot{\mu}} \partial_{\dot{\beta}} b^{\dot{\mu}} \right) - \frac{1}{2} \epsilon_{\alpha\beta} F_{\dot{\mu}\dot{\nu}} F^{\alpha\dot{\mu}} F^{\beta\dot{\nu}} \\ &\quad \left. + F^{\dot{\mu}\dot{\nu}} F_{\dot{\lambda}\dot{\nu}} \partial_{\dot{\mu}} b^{\dot{\lambda}} + F_{\alpha\dot{\mu}} F^{\alpha}_{\dot{\nu}} \partial^{\dot{\mu}} b^{\dot{\nu}} - F_{\alpha\dot{\mu}} \partial^{\alpha} b_{\dot{\nu}} F^{\dot{\mu}\dot{\nu}} \right\}. \end{aligned} \quad (83)$$

In order to integrate out $H_{i\dot{2}\dot{3}}$, note that we can impose the gauge fixing condition

$$\epsilon^{\dot{\mu}\dot{\nu}\dot{\lambda}} \partial_{\dot{\mu}} b_{\dot{\nu}} = 0, \quad (84)$$

so that

$$b_{\dot{\mu}} = \partial_{\dot{\mu}} c \quad (85)$$

for some function c . Solving c from

$$H_{i\dot{2}\dot{3}} = \partial_{\dot{\mu}} b^{\dot{\mu}}, \quad (86)$$

we find

$$b^{\dot{\mu}} = \partial^{\dot{\mu}} \dot{\partial}^{-2} H_{i\dot{2}\dot{3}}, \quad (87)$$

where $\dot{\partial}^{-2}$ is the inverse operator of the Laplacian $\dot{\partial}^2 \equiv \partial_{\dot{\mu}} \partial^{\dot{\mu}}$. Denoting the Green's function of the Laplacian by G so that

$$\dot{\partial}^2 G(y - y') = \delta^{(3)}(y - y'), \quad (88)$$

⁹ Since the 3 gauge transformation parameters $\kappa^{\dot{\mu}}$ are subject to the condition $\partial_{\dot{\mu}} \kappa^{\dot{\mu}} = 0$, there are only 2 functionally independent degrees of freedom in $\kappa^{\dot{\mu}}$.

where y and y' represent the coordinates in the directions y^1, y^2, y^3 . We have

$$\dot{\partial}^{-2}\phi(y) = \int d^3y' G(y-y')\phi(y). \quad (89)$$

Plugging (87) into the action, we get an action as a functional of $H_{i\dot{2}\dot{3}}$ and a_A . To the first order in g , we can integrate out $H_{i\dot{2}\dot{3}}$ and the action becomes

$$S''_{gauge}[a_A] = \int d^2x d^3y \left\{ -\frac{1}{4}F_{AB}F^{AB} + g \left[-F_{01}\mathcal{C} - \frac{1}{2}\epsilon_{\alpha\beta}F_{\dot{\mu}\dot{\nu}}F^{\alpha\dot{\mu}}F^{\beta\dot{\nu}} \right. \right. \\ \left. \left. - F^{\dot{\mu}\dot{\nu}}F_{\dot{\lambda}\dot{\nu}}\partial_{\dot{\mu}}\partial^{\dot{\lambda}}\dot{\partial}^{-2}F_{01} - F_{\alpha\dot{\mu}}F^{\alpha}_{\dot{\nu}}\partial^{\dot{\mu}}\partial^{\dot{\nu}}\dot{\partial}^{-2}F_{01} + F_{\alpha\dot{\mu}}F^{\dot{\mu}\dot{\nu}}\partial^{\alpha}\partial_{\dot{\nu}}\dot{\partial}^{-2}F_{01} \right] \right\} \quad (90)$$

where

$$\mathcal{C} = -\frac{1}{2}F_{\dot{\mu}\dot{\nu}}F^{\dot{\mu}\dot{\nu}} - \epsilon^{\alpha\beta}F_{\alpha\dot{\mu}}\partial_{\beta}\partial^{\dot{\mu}}\dot{\partial}^{-2}F_{01}. \quad (91)$$

Apparently the action becomes nonlocal at order $\mathcal{O}(g)$.

In principle, using (87) to rewrite the action as a functional of a_A and $H_{i\dot{2}\dot{3}}$, we can integrate out $H_{i\dot{2}\dot{3}}$ to an arbitrary order in g . The resulting action would be a functional of F_{AB} with higher derivatives and $\dot{\partial}^{-2}$.

6 Matter Fields

In the above we have ignored the matter fields in the NP M5-brane theory. It is straightforward to repeat the derivations above with the matter fields included. Analogous to (49), we get

$$S^{(4)}[b^{\dot{\mu}}, a_A, \check{B}_{\alpha}^{\dot{\mu}}, X^i, \Psi] = \int d^2x d^3y \left\{ -\frac{1}{2}\mathcal{D}_{\dot{\mu}}X^i\mathcal{D}^{\dot{\mu}}X^i - \frac{1}{2}\partial_{\alpha}X^i\partial^{\alpha}X^i + g\check{B}_{\alpha}^{\dot{\mu}}\partial_{\dot{\mu}}X^i\partial^{\alpha}X^i \right. \\ - \frac{g^2}{2}\check{B}_{\alpha}^{\dot{\mu}}\check{B}_{\dot{\nu}}^{\alpha}\partial_{\dot{\mu}}X^i\partial^{\dot{\nu}}X^i - \frac{g^2}{8}\epsilon^{\dot{\mu}\dot{\rho}\dot{\tau}}\epsilon_{\dot{\nu}\dot{\sigma}\dot{\delta}}F_{\dot{\rho}\dot{\tau}}F^{\dot{\sigma}\dot{\delta}}\partial_{\dot{\mu}}X^i\partial^{\dot{\nu}}X^i \\ - \frac{g^4}{4}\{X^{\dot{\mu}}, X^i, X^j\}^2 - \frac{g^4}{12}\{X^i, X^j, X^k\}^2 \\ + \frac{i}{2}\bar{\Psi}\Gamma^{\alpha}\partial_{\alpha}\Psi + \frac{i}{2}\bar{\Psi}\Gamma^{\dot{\rho}}\mathcal{D}_{\dot{\rho}}\Psi + g\frac{i}{4}\bar{\Psi}\Gamma^2\epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}}F_{\dot{\nu}\dot{\rho}}\partial_{\dot{\mu}}\Psi - g\frac{i}{2}\bar{\Psi}\Gamma^{\alpha}\check{B}_{\alpha}^{\dot{\mu}}\partial_{\dot{\mu}}\Psi \\ + g^2\frac{i}{2}\bar{\Psi}\Gamma_{\dot{\mu}i}\{X^{\dot{\mu}}, X^i, \Psi\} + g^2\frac{i}{4}\bar{\Psi}\Gamma_{ij}\Gamma_{i\dot{2}\dot{3}}\{X^i, X^j, \Psi\} \\ - \frac{1}{2g^2} - \frac{1}{2}(\mathcal{H}_{i\dot{2}\dot{3}})^2 - \frac{1}{4}\mathcal{F}_{\dot{\nu}\dot{\rho}}\mathcal{F}^{\dot{\nu}\dot{\rho}} - \frac{1}{4}(\epsilon_{\dot{\mu}\dot{\nu}\dot{\rho}}(\partial_{\alpha}b^{\dot{\mu}} - V_{\dot{\sigma}}^{\dot{\mu}}\check{B}_{\alpha}^{\dot{\sigma}}))^2 \\ \left. + \epsilon^{\alpha\beta}F_{\beta\dot{\mu}}\check{B}_{\alpha}^{\dot{\mu}} + \frac{g}{2}\epsilon^{\alpha\beta}F_{\dot{\mu}\dot{\nu}}\check{B}_{\alpha}^{\dot{\mu}}\check{B}_{\beta}^{\dot{\nu}} \right\}. \quad (92)$$

With the matter fields included, the action is still no more than quadratic in $\check{B}_\alpha^{\dot{\mu}}$ and so we can still integrate it out. This is equivalent to solving the equation of motion for $\check{B}_\alpha^{\dot{\mu}}$ and plugging it back into the action. The new equation of motion for $\check{B}_\alpha^{\dot{\mu}}$ is

$$V_{\dot{\mu}}^{\dot{\nu}}(\partial^\alpha b_{\dot{\nu}} - V_{\dot{\nu}}^{\dot{\rho}} \check{B}_{\dot{\rho}}^\alpha) + \epsilon^{\alpha\beta} F_{\beta\dot{\mu}} + g\epsilon^{\alpha\beta} F_{\dot{\mu}\dot{\nu}} \check{B}_\beta^{\dot{\nu}} + g\partial_{\dot{\mu}} X^i \partial^\alpha X^i - g\frac{i}{2} \bar{\Psi} \Gamma^\alpha \partial_{\dot{\mu}} \Psi - g^2 \check{B}_{\dot{\nu}}^\alpha \partial_{\dot{\mu}} X^i \partial^{\dot{\nu}} X^i = 0. \quad (93)$$

Its solution is

$$\begin{aligned} \hat{B}_\alpha^{\dot{\mu}} &= (\mathbf{M}^{-1})^{\dot{\mu}\dot{\nu}}_{\alpha\beta} (V_{\dot{\nu}}^{\dot{\rho}} \partial^\beta b_{\dot{\rho}} + \epsilon^{\beta\gamma} F_{\gamma\dot{\nu}} + g\partial_{\dot{\nu}} X^i \partial^\beta X^i - g\frac{i}{2} \bar{\Psi} \Gamma^\beta \partial_{\dot{\nu}} \Psi) \\ &\equiv (\mathbf{M}^{-1})^{\dot{\mu}\dot{\nu}}_{\alpha\beta} W_{\dot{\nu}}^\beta, \end{aligned} \quad (94)$$

where

$$\mathbf{M}_{\dot{\mu}\dot{\nu}}^{\alpha\beta} \equiv (V_{\dot{\mu}\dot{\rho}} V_{\dot{\nu}}^{\dot{\rho}} + g^2 \partial_{\dot{\mu}} X^i \partial_{\dot{\nu}} X^i) \delta^{\alpha\beta} - g\epsilon^{\alpha\beta} F_{\dot{\mu}\dot{\nu}}, \quad (95)$$

and $(\mathbf{M}^{-1})^{\dot{\mu}\dot{\nu}}_{\alpha\beta}$ is defined by

$$(\mathbf{M}^{-1})^{\dot{\lambda}\dot{\mu}}_{\gamma\alpha} \mathbf{M}_{\dot{\mu}\dot{\nu}}^{\alpha\beta} = \delta^{\dot{\lambda}}_{\dot{\nu}} \delta_\gamma^\beta. \quad (96)$$

Finally, we get the action

$$\begin{aligned} S'_{gauge}[b^\mu, a_A, X^i, \Psi] &= \int d^2 x d^3 y \left\{ -\frac{1}{2} \mathcal{D}_\mu X^i \mathcal{D}^\mu X^i - \frac{1}{2} \partial_\alpha X^i \partial^\alpha X^i \right. \\ &\quad - \frac{g^2}{8} \epsilon^{\dot{\mu}\dot{\rho}\dot{\tau}} \epsilon_{\dot{\nu}\dot{\sigma}\dot{\delta}} F_{\dot{\rho}\dot{\tau}} F^{\dot{\sigma}\dot{\delta}} \partial_{\dot{\mu}} X^i \partial^{\dot{\nu}} X^i \\ &\quad - \frac{g^4}{4} \{X^\mu, X^i, X^j\}^2 - \frac{g^4}{12} \{X^i, X^j, X^k\}^2 \\ &\quad + \frac{i}{2} \bar{\Psi} \Gamma^\alpha \partial_\alpha \Psi + \frac{i}{2} \bar{\Psi} \Gamma^{\dot{\rho}} \mathcal{D}_{\dot{\rho}} \Psi + g\frac{i}{4} \bar{\Psi} \Gamma^2 \epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}} F_{\dot{\nu}\dot{\rho}} \partial_{\dot{\mu}} \Psi \\ &\quad + g^2 \frac{i}{2} \bar{\Psi} \Gamma_{\dot{\mu}\dot{i}} \{X^\mu, X^i, \Psi\} + g^2 \frac{i}{4} \bar{\Psi} \Gamma_{ij} \Gamma_{\dot{i}\dot{j}} \{X^i, X^j, \Psi\} \\ &\quad \left. - \frac{1}{2g^2} - \frac{1}{2} (\mathcal{H}_{\dot{i}\dot{j}\dot{k}})^2 - \frac{1}{4} \mathcal{F}_{\dot{\nu}\dot{\rho}} \mathcal{F}^{\dot{\nu}\dot{\rho}} + \frac{1}{2} W_{\dot{\mu}}^\alpha (\mathbf{M}^{-1})^{\dot{\mu}\dot{\nu}}_{\alpha\beta} W_{\dot{\nu}}^\beta \right\}. \quad (97) \end{aligned}$$

Here the fields \mathcal{F} are defined by the same expressions as before but with the new definition of \hat{B} .

At the 0-th order of g , the action is just

$$S''^{(0)}_{gauge}[a_A, X^i, \Psi] \simeq \int d^2 x d^3 y \left\{ -\frac{1}{4} F_{AB} F^{AB} - \frac{1}{2} \partial_A X^i \partial^A X^i + \frac{i}{2} \bar{\Psi} \Gamma^A \partial_A \Psi \right\} \quad (98)$$

after we integrate out the VPD gauge fields. This defines a Maxwell's theory with neutral bosons X and fermions Ψ .

For completeness let us also give the expression of the action to the 1st order:

$$\begin{aligned}
S'_{gauge}[b^\mu, a_A, X^i, \Psi] \simeq & \int d^2x d^3y \left\{ -\frac{1}{2} \partial_\alpha X^i \partial^\alpha X^i - \frac{1}{2} \partial_{\dot{\mu}} X^i \partial^{\dot{\mu}} X^i + \frac{i}{2} \bar{\Psi} \Gamma^\alpha \partial_\alpha \Psi + \frac{i}{2} \bar{\Psi} \Gamma^{\dot{\mu}} \partial_{\dot{\mu}} \Psi \right. \\
& + g \epsilon^{\alpha\beta} F_{\beta\dot{\mu}} \partial^{\dot{\mu}} X^i \partial_\alpha X^i + g \partial^{\dot{\mu}} X^i \partial_\alpha X^i \partial^\alpha b_{\dot{\mu}} - g \frac{i}{2} \epsilon^{\alpha\beta} F_{\beta\dot{\mu}} \bar{\Psi} \Gamma_\alpha \partial^{\dot{\mu}} \Psi \\
& - g \frac{i}{2} \bar{\Psi} \Gamma_\alpha \partial^{\dot{\mu}} \Psi \partial^\alpha b_{\dot{\mu}} - g \partial_{\dot{\mu}} X^i \partial^{\dot{\mu}} X^i \partial_{\dot{\rho}} b^{\dot{\rho}} + g \partial^{\dot{\mu}} X^i \partial_{\dot{\rho}} X^i \partial_{\dot{\mu}} b^{\dot{\rho}} \\
& + g \frac{i}{2} \bar{\Psi} \Gamma^{\dot{\rho}} \partial_{\dot{\rho}} \Psi \partial_{\dot{\nu}} b^{\dot{\nu}} - g \frac{i}{2} \bar{\Psi} \Gamma^{\dot{\rho}} \partial_{\dot{\nu}} \Psi \partial_{\dot{\rho}} b^{\dot{\nu}} + g \frac{i}{4} \bar{\Psi} \Gamma^2 \epsilon^{\dot{\mu}\dot{\nu}\dot{\rho}} F_{\dot{\nu}\dot{\rho}} \partial_{\dot{\mu}} \Psi \\
& - \frac{1}{2} \mathcal{H}_{i\dot{1}\dot{2}\dot{3}} \mathcal{H}^{i\dot{2}\dot{3}} - \frac{1}{4} \mathcal{F}_{\dot{\nu}\dot{\rho}} \mathcal{F}^{\dot{\nu}\dot{\rho}} - \frac{1}{2} F_{\beta\dot{\mu}} F^{\beta\dot{\mu}} - \frac{1}{2} \epsilon^{\alpha\beta} F_{\alpha\beta} \partial_{\dot{\mu}} b^{\dot{\mu}} \\
& - g \epsilon^{\alpha\beta} F_{\beta\dot{\mu}} \partial_\alpha b_{\dot{\nu}} \partial^{\dot{\nu}} b^{\dot{\mu}} + \frac{1}{2} g \epsilon^{\alpha\beta} F_{\dot{\mu}\dot{\nu}} \partial_\alpha b^{\dot{\mu}} \partial_{\dot{\beta}} b^{\dot{\nu}} + g F_{\dot{\mu}\dot{\nu}} F^{\alpha\dot{\nu}} \partial_\alpha b^{\dot{\mu}} \\
& \left. + g F^{\alpha\dot{\nu}} F_{\alpha\dot{\mu}} \partial_{\dot{\nu}} b^{\dot{\mu}} + \frac{1}{2} g \epsilon^{\alpha\beta} F_{\beta\dot{\mu}} F^{\dot{\mu}\dot{\nu}} F_{\alpha\dot{\nu}} + O(g^2) \right\}. \tag{99}
\end{aligned}$$

The full action (97) inherits the full supersymmetry from the NP M5-brane theory because DDR preserves global SUSY, and duality transformation is an equivalence relation. Nevertheless it is not totally trivial to derive the explicit SUSY transformation rules for all the variables, in particular those arise as Lagrange multipliers. We leave this for future study.

7 Generalization to Multiple Dp-branes

To generalize the story about a single D4-brane to a system of multiple Dp-branes, we notice first that the VPD for a volume $(p-1)$ -form is generated by a $(p-1)$ -bracket

$$\{f_1, f_2, \dots, f_{p-1}\} \equiv \epsilon^{\dot{\mu}_1 \dot{\mu}_2 \dots \dot{\mu}_{p-1}} \partial_{\dot{\mu}_1} f_1 \partial_{\dot{\mu}_2} f_2 \dots \partial_{\dot{\mu}_{p-1}} f_{p-1}. \tag{100}$$

We define a $(p-2)$ -form gauge potential $b_{\dot{\mu}_1 \dots \dot{\mu}_{p-2}}$ and its dual

$$b^{\dot{\mu}_1} = \frac{1}{(p-2)!} \epsilon^{\dot{\mu}_1 \dot{\mu}_2 \dots \dot{\mu}_{p-1}} b_{\dot{\mu}_2 \dots \dot{\mu}_{p-1}}. \tag{101}$$

Let

$$X^{\dot{\mu}} = \frac{y^{\dot{\mu}}}{g} + b^{\dot{\mu}} \tag{102}$$

and the field strength \mathcal{H} can be defined as

$$\mathcal{H}_{\dot{\mu}_1 \dot{\mu}_2 \dots \dot{\mu}_{p-1}} \equiv g^{p-2} \{X^{\dot{\mu}_1}, X^{\dot{\mu}_2}, \dots, X^{\dot{\mu}_{p-1}}\} - \frac{1}{g} = \partial_{\dot{\mu}} b^{\dot{\mu}} + \mathcal{O}(g). \tag{103}$$

In terms of b^μ the gauge transformation is exactly of the same form as (56), and the parameter κ^μ is still divergenceless. The only change is that the range of the indices $\dot{\mu}, \dot{\nu}$ becomes $2, 3, \dots, p$.¹⁰

While we do not intend to promote the VPD gauge potential b^μ to a matrix mostly because we do not know how to modify its gauge transformation law, we shall replace the $U(1)$ potential by a $U(N)$ potential a_A , which is now an $N \times N$ anti-Hermitian matrix of 1-forms. The $U(N)$ gauge transformation of a_A should be defined by

$$\delta a_A = [D_A, \lambda] + g(\kappa^\mu \partial_\mu a_A + a_\mu \partial_A \kappa^\mu), \quad (104)$$

where $D_A \equiv \partial_A + a_A$. It modifies (61) only by replacing $\partial_A \lambda$ by $[D_A, \lambda]$. The gauge transformation parameter λ is an $N \times N$ anti-Hermitian matrix but κ^μ is 1×1 . The range of the index A is now $A = 0, 1, 2, \dots, p$. Decomposing the potential a_A into the $U(1)$ part and the $SU(N)$ part

$$a_A = a_A^{U(1)} + a_A^{SU(N)}, \quad (105)$$

the gauge transformation of $a_A^{U(1)}$ is exactly the same as before (61).

We can define $V_{\dot{\mu}}^{\dot{\nu}}$ and $\hat{B}_\alpha^{\dot{\mu}}$ using the same expressions (44)–(51) as before

$$V_{\dot{\nu}}^{\dot{\mu}} \equiv \delta_{\dot{\nu}}^{\dot{\mu}} + g \partial_{\dot{\nu}} b^{\dot{\mu}}, \quad (106)$$

$$M_{\dot{\mu}\dot{\nu}}^{\alpha\beta} \equiv V_{\dot{\mu}\dot{\rho}} V_{\dot{\nu}}^{\dot{\rho}} \delta^{\alpha\beta} - g \epsilon^{\alpha\beta} F_{\dot{\mu}\dot{\nu}}^{U(1)}, \quad (107)$$

$$\hat{B}_\alpha^{\dot{\mu}} \equiv (M^{-1})^{\dot{\mu}\dot{\nu}}_{\alpha\beta} (V_{\dot{\nu}}^{\dot{\sigma}} \partial^{\dot{\sigma}} b_{\dot{\sigma}} + \epsilon^{\beta\gamma} F_{\gamma\dot{\nu}}^{U(1)}), \quad (108)$$

but with the field strength $F_{\dot{\mu}\dot{\nu}}^{U(1)}$ being the $U(1)$ part of the $U(N)$ field strength, so that their gauge transformations remain the same. The range of the indices α, β is still $0, 1$.

The naive definition of field strength $F_{AB} \equiv [D_A, D_B]$ is not covariant. They transform like

$$\delta F_{AB} = [F_{AB}, \lambda] + g \kappa^\mu \partial_\mu F_{AB} + g[(\partial_A \kappa^\mu) F_{\mu B} - (\partial_B \kappa^\mu) F_{\mu A}]. \quad (109)$$

It turns out that exactly the same expressions as (69)–(71) give the covariant field strengths. For the convenience of the reader we reproduce them here

$$\begin{aligned} \mathcal{F}_{\dot{\mu}\dot{\nu}} &= F_{\dot{\mu}\dot{\nu}} + g[\partial_{\dot{\sigma}} b^{\dot{\sigma}} F_{\dot{\mu}\dot{\nu}} - \partial_{\dot{\mu}} b^{\dot{\sigma}} F_{\dot{\sigma}\dot{\nu}} - \partial_{\dot{\nu}} b^{\dot{\sigma}} F_{\dot{\mu}\dot{\sigma}}] \\ &= V_{\dot{\rho}}^{\dot{\rho}} F_{\dot{\mu}\dot{\nu}} + V_{\dot{\mu}}^{\dot{\rho}} F_{\dot{\rho}\dot{\nu}} + V_{\dot{\nu}}^{\dot{\rho}} F_{\dot{\rho}\dot{\mu}}, \end{aligned} \quad (110)$$

$$\mathcal{F}_{\alpha\dot{\mu}} = V_{\dot{\mu}}^{-1}{}^{\dot{\nu}} (F_{\alpha\dot{\nu}} + g F_{\dot{\nu}\dot{\delta}} \hat{B}_\alpha^{\dot{\delta}}), \quad (111)$$

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + g[-F_{\alpha\dot{\mu}} \hat{B}_\beta^{\dot{\mu}} - F_{\dot{\mu}\beta} \hat{B}_\alpha^{\dot{\mu}} + g F_{\dot{\mu}\dot{\nu}} \hat{B}_\alpha^{\dot{\mu}} \hat{B}_\beta^{\dot{\nu}}]. \quad (112)$$

¹⁰ The indices $2, 3, \dots$ would be denoted as $\dot{1}, \dot{2}, \dots$ in previous sections.

They transform like

$$\delta \mathcal{F}_{AB} = [\mathcal{F}_{AB}, \lambda - g \kappa^\mu \partial_\mu]. \quad (113)$$

From this expression it is easy to check that the gauge symmetry algebra is given by

$$[\delta_1, \delta_2] = \delta_3, \quad (114)$$

where δ_i is the gauge transformation with parameters λ_i, κ_i^μ and

$$\lambda_3 = [\lambda_1, \lambda_2] + g(\kappa_2^\mu \partial_\mu \lambda_1 - \kappa_1^\mu \partial_\mu \lambda_2), \quad (115)$$

$$\kappa_3^\mu = g(\kappa_2^\nu \partial_\nu \kappa_1^\mu - \kappa_1^\nu \partial_\nu \kappa_2^\mu). \quad (116)$$

In view of the D4-brane action (73), it is now natural to define the action for the gauge fields on multiple Dp-branes in R-R $(p-1)$ -form field background as

$$\begin{aligned} S_{gauge}^{Dp}[b^\mu, a_A] = & \int d^2 x d^{p-1} y \left\{ -\frac{1}{2} \frac{1}{(p-1)!} \mathcal{H}_{\dot{\mu}_1 \dots \dot{\mu}_{p-1}} \mathcal{H}^{\dot{\mu}_1 \dots \dot{\mu}_{p-1}} + \frac{1}{2g} \epsilon^{\alpha\beta} \mathcal{F}_{\alpha\beta}^{U(1)} \right. \\ & \left. - \frac{1}{4} \mathcal{F}_{\dot{\nu}\dot{\rho}}^{U(1)} \mathcal{F}_{U(1)}^{\dot{\nu}\dot{\rho}} + \frac{1}{2} \mathcal{F}_{\beta\dot{\mu}}^{U(1)} \mathcal{F}_{U(1)}^{\beta\dot{\mu}} - \frac{1}{4} \text{tr} \left(\mathcal{F}_{AB}^{SU(N)} \mathcal{F}_{SU(N)}^{AB} \right) \right\}. \end{aligned} \quad (117)$$

If we focus our attention on the $U(1)$ part of the 1-form gauge potential a_A and the VPD gauge potential b^μ , everything is exactly the same as before. The VPD field strength \mathcal{H} is dual to only the $U(1)$ part of \mathcal{F}_{01} . But since the $SU(N)$ part of the field strength \mathcal{F}_{AB} involves the VPD potential b^μ , the $U(1)$ part of a_A couples to the $SU(N)$ part indirectly through b^μ . This is different from the usual Yang-Mills theory of $U(N)$ gauge symmetry, for which the $U(1)$ part decouples, but similar to the noncommutative $U(N)$ YM theory.

To the 0-th order in g , the action is

$$S'^{Dp(0)}_{gauge}[b^\mu, a_A] = \int d^2 x d^3 y \left\{ -\frac{1}{2} (H_{23\dots p} + F_{01}^{U(1)})^2 - \frac{1}{4} F_{AB}^{U(1)} F_{U(1)}^{AB} - \frac{1}{4} \text{tr} \left(F_{AB}^{SU(N)} F_{SU(N)}^{AB} \right) \right\}, \quad (118)$$

where $H_{23\dots p} = \partial_\mu b^\mu$. Again, since $H_{23\dots p}$ is the only component of the field strength for the gauge potential b^μ , we can integrate it out and the action reduces to that of a Yang-Mills theory in $(p+1)$ dimensions.

In fact, the VPD symmetry allows us to impose the gauge fixing condition

$$\partial^{\dot{\mu}_1} b_{\dot{\mu}_1 \dot{\mu}_2 \dots \dot{\mu}_{p-2}} = 0 \quad \Leftrightarrow \quad \partial^\mu b^\nu - \partial^\nu b^\mu = 0. \quad (119)$$

This condition allows us to solve b^μ in terms of $H_{23\dots p}$ as

$$b^\mu = \partial^\mu \dot{\partial}^{-2} H_{23\dots p}, \quad (120)$$

where $\dot{\partial}^{-2}$ is the inverse of the Laplace operator $\dot{\partial}^2 \equiv \partial_\mu \partial^\mu$. Like what we did in Sec. 5.2, we can continue to integrate out $H_{23\dots p}$ at higher orders of g using the relation (120) for every term involving b^μ in the action, although we would get a nonlocal action in the end. In principle we can write down a nonlocal action of a_A without any trace of b^μ or $H_{23\dots p}$ as an expansion of g to an arbitrary order.

8 Conclusion and Discussion

In this paper, we showed that the C -field background induces a Nambu-Poisson structure on the D4-brane which generates the gauge symmetry of volume-preserving diffeomorphisms. The potential b^μ for this new gauge symmetry is the electric-magnetic dual of the $U(1)$ gauge potential a_α in the D4-brane theory. In the limit $g \rightarrow 0$ the NP D4-brane theory reduces to the usual Maxwell theory for a D4-brane. Our D4-brane action in C -field background is a good approximation of the system in the limit (35) and (36).

It is possible that the NP M5-brane theory is good at low energy for a wider range of limits than that given in (2). It may be possible that it is a good effective theory when $g_{22} \sim \epsilon^{2a}$ for some positive real number a within a certain range, while all other variables take the same limit as before. This would allow us to impose the condition (36) simultaneously with (35). The compactification radius R in the double dimensional reduction scales like

$$R \sim \epsilon^a, \quad (121)$$

if $g_{22} \sim \epsilon^{2a}$. The type IIA parameters then scale like

$$\ell_s \sim \epsilon^{(1-a)/2}, \quad g_s \sim \epsilon^{(3a-1)/2}, \quad (122)$$

while $2\pi\alpha' B_{01}$ is still finite when $\epsilon \rightarrow 0$. If

$$1/3 < a < 1, \quad (123)$$

we would have $\ell_s, g_s, R/\ell_s, R/\ell_P \rightarrow 0$ at the same time.

In the previous section we have partially generalized the NP D4-brane theory to theories for Dp -branes in constant $(p-1)$ -form background, although we have not yet included matter fields. In the gauge field sector of the Dp -brane theory, there is a $(p-2)$ -form gauge potential b for the volume-preserving diffeomorphism symmetry in the $(p-1)$ directions selected by the R-R $(p-1)$ -form background. The volume-preserving diffeomorphism is generated by a generalization of the Nambu-Poisson bracket with $(p-1)$

slots. The field strength \mathcal{H} of the potential b^μ is dual to the $U(1)$ field strength in the sense that, to the lowest order in g ,

$$F^{01} = \frac{1}{(p-1)!} \epsilon^{01\dot{\mu}_1 \dots \dot{\mu}_{p-1}} H_{\dot{\mu}_1 \dots \dot{\mu}_{p-1}}, \quad (124)$$

so there is no independent physical degrees of freedom in b^μ apart from those in the $U(1)$ gauge field a . The all-order relation between the covariant field strength \mathcal{H} and \mathcal{F} is very complicated because the volume preserving diffeomorphism is non-Abelian. Both the definitions of \mathcal{H} and \mathcal{F} are quite non-trivial. Apart from the advancement in our understanding about D-branes in string theory, the novelty of the structure of gauge symmetry is intriguing by itself. This is an efficient gauge theory in which the gauge potentials for two gauge symmetries share the same physical degrees of freedom.

If we take T-duality along the x^1 -direction of the D4-brane considered in this paper, we get a D3-brane in an R-R 4-form potential background. The background 4-form can be decomposed as the wedge product of a 1-form in the direction \tilde{x}^1 T-dual to x^1 and the 3-form C along the D3-brane worldvolume. The 3-form C defines a volume-form and the corresponding VPD is inherited from the D4-brane. We still need the gauge fields b^μ for D3. The connection between F_{01} and $H_{\dot{1}\dot{2}\dot{3}}$ at the 0-th order for a D4-brane becomes the connection between \tilde{p}_1 , the momentum in the direction of \tilde{x}^1 , and $H_{\dot{1}\dot{2}\dot{3}}$. Extending this conclusion to Dp -branes, we claim that for a Dp -brane in the RR $(p+1)$ -form potential background

$$D^{(p+1)} = V^{(1)} \wedge C^{(p)}, \quad (125)$$

where $V^{(1)}$ is transverse and $C^{(p)}$ is parallel to the Dp -brane, the VPD corresponding to the volume-form $C^{(p)}$ shares the same gauge field degrees of freedom with the component of the momentum p in the direction of $V^{(1)}$.

For future works, we would like to include matter fields for Dp -branes in large R-R $(p-1)$ -form gauge potential background. It will also be important to find the explicit expressions of SUSY transformation laws.

As a final goal of this line of research, one would like to generalize the results to the ultimate generality of multiple Dp -branes and NS 5-branes in all combinations NS-NS and R-R field background in various limits. An immediate challenging problem is to define a deformation of VPD such that it is the electric magnetic dual of the noncommutative and/or non-Abelian gauge symmetry. There are related No-Go theorems [24, 25] suggesting that there will be brand new gauge symmetries yet to be discovered.

Acknowledgment

The authors thank Chien-Ho Chen, Wei-Ming Chen, Chong-Sun Chu, Kazuyuki Furuchi, Petr Hořava, Kuo-Wei Huang, Yu-tin Huang, Hirotaka Irie, Hiroshi Isono, Sheng-Lan Ko, Yutaka Matsuo, Yu Nakayama, Hiroshi Ooguri, Ryu Sasaki, John Schwarz, Tomohisa Takimi, Wen-Yu Wen and Chen-Pin Yeh for helpful discussions. The final stage of this work was completed during a visit of P.M.H. at CalTech. He thanks the hospitality of the high energy theory group at CalTech. This work is supported in part by the National Science Council, the NSC internship program, the National Center for Theoretical Sciences, and the Center for Theoretical Sciences at National Taiwan University.

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